Dynamic Game Model of Endogenous Growth with Consumption Externalities*

K. Hori† and A. Shibata‡

Abstract. This paper introduces consumption externalities into an endogenous growth model of common capital accumulation and characterizes balanced growth equilibria. Contrary to the standard argument in previous studies, we show that the growth rate in a feedback Nash equilibrium can be higher than that in an open-loop Nash equilibrium if agents strongly admire the consumption of others. This result is irrelevant to whether preferences exhibit "keeping up with the Joneses" or "running away from the Joneses".

Key Words. Differential game, Consumption externalities, Endogenous growth, Open-loop Nash equilibrium, Feedback Nash equilibrium

1 Introduction

In the literature on commons games, it is usually argued that the lack of agents' commitment to their future actions leads to the so-called tragedy of the commons; that is, if agents condition their actions on the basis of the aggregate stock of the commons, rather than committing themselves to initial decisions that depends only on the aggregate stock of the commons, then this could worsen the overconsumption and/or underinvestment of the commons. For example, Gordon (Ref. 1) is the first study that presents an example of the tragedy of the commons. Fershtman and Nitzan (Ref. 2) and Levhari and Mirman (Ref. 3) develop

*The authors acknowledge K. Futagami and T. Kunieda for their encouragement and helpful suggestions. They also appreciate G. Leitmann, A. Miele, and anonymous referees for helpful comments. This work was supported by the Global Center of Excellence Program of the Ministry of Education, Culture, Sports, Science and Technology.

†Researcher, Institute of Economic Research, Kyoto University, Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501, Japan

‡Professor, Institute of Economic Research, Kyoto University, Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501, Japan

a dynamic model of the voluntary provision of public goods and show that conditioning each agent's contribution on the collective contributions aggravates the degree of the free-rider problem. Using endogenous growth models, Tornell and Velasco (Ref. 4) and Shibata (Ref. 5) show that the balanced growth rate without commitment is lower than that with commitment. Moreover, Benhabib and Radner (Ref. 6), Cozzi (Ref. 7), Dockner and Sorger (Ref. 8), Sorger (Ref. 9), Vencatachellum (Ref. 10), Vencatachellum (Ref. 11), Dockner and Nishimura (Ref. 12), and Luckraz (Ref. 13) study the other aspects of equilibria in similar situations. However, contrary to the literature, our daily experience and observation tell us that monitoring behavior mitigates the free-rider problem: e.g., the behavior of agents conditioned on natural resource investigations tends to raise the growth rates of natural resource stocks. Therefore, this paper develops a simple dynamic game model of common capital accumulation with consumption externalities that can explain our daily experience and observation.

The existence of consumption externalities is emphasized in an early work by Veblen (Ref. 14) and validated as a determinant of aggregate consumption by Duesenberry (Ref. 15). Moreover, many recent studies analyze the effects of consumption externalities. For example, to reconcile the equity premium puzzle, Abel (Ref. 16), Constantinides (Ref. 17), and Galí (Ref. 18) incorporate the consumption externalities into consumption-based asset pricing models. Liu and Turnovsky (Ref. 19), Ljungqvist and Uhlig (Ref. 20), Turnovsky and Monteiro (Ref. 21), and Mino (Ref. 22) construct growth models with consumption externalities. In addition to these theoretical and calibration studies, there is much empirical evidence showing the importance of consumption externalities in the real world. Kosicki (Ref. 23) finds that an individual's regional income rank significantly affects his consumption behavior; Clark and Oswald (Ref. 24) find that workers' satisfaction levels negatively depend on the relative wage rates. Using the data of the United States, Europe, and Japan, Easterlin (Ref. 25) finds supportive evidence of the existence of consumption externalities. Moreover, based on experiments, Kagel, Kim, and Moser (Ref. 26) and Zizzo and Oswald (Ref. 27) find that economic agents' utilities depend on their relative income levels. Along this line of research, we introduce consumption externalities into a dynamic game model of endogenous growth and analyze how the introduction of the externalities modifies the standard results on the growth rates with and without commitment.

Although there are very few studies that investigate the role of consumption externalities in the frameworks of the dynamic game theory, Wirl and Feichtinger (Ref. 28) and Futagami and Shinkai (Ref. 29) are exceptions. They develop differential game models with consumption externalities and examine the effects of the consumption externalities on equilibrium dynamics in a similar way to ours. However, our model is significantly different from theirs and more general than theirs. Dupor and Liu (Ref. 30) classify the effects of consumption externalities into four cases: jealousy, admiration, "keeping up with the Joneses" (KUJ), and "running away from the Joneses" (RAJ). Since Wirl and Feichtinger (Ref. 28) use a utility function that is additively separable between the agent's and others' consumption, their model inherently excludes the KUJ/RAJ characteristics of consumption externalities. Moreover, their model is not an endogenous growth one. Futagami and Shinkai (Ref. 29) construct a similar endogenous growth model to ours. However, in their specification of utility, admiration is always equivalent to KUJ and jealousy is equivalent to RAJ, and thus only two cases can be analyzed. In contrast to these studies, our model distinguishes the four types of consumption externalities and can analyze their implication on the economic growth rate.

2 Model

In this paper, we use only the scalar-valued functions, parameters, and variables. There are N homogenous agents in our economy. They jointly produce a good by using common capital and divide it into consumption and common capital accumulation. The lifetime utility function of each agent is assumed to be additively separable in time. The agent subjective discount rate is denoted by ρ and the elasticity of intertemporal substitution is represented by η. Therefore, the problem of agent i is to maximize the following lifetime utility:

$$\int_0^\infty u_i \exp(-\rho t)\,\mathrm{d}t, \quad i = 1, \ldots, N. \tag{1}$$

Here, u_i denotes the instantaneous utility function and is specified as

$$u_i = \frac{\eta}{\eta - 1}\left(c_i \overline{c}_i^{-\alpha}\right)^{1-\frac{1}{\eta}}, \quad i = 1, \ldots, N,$$

where $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable; $\overline{c}_i = \sum_{j \neq i} c_j/(N-1)$; $\rho, \eta \in \mathbb{R}_{++}$; and $\alpha \in (-\infty, 1)$. Here, c_i is the consumption of agent i and $\overline{c}_i$ is the level of average consumption of other agents, which represents the externalities from other agents' consumption. The parameter α represents the magnitude of the external effects of consumption. Following Dupor and Liu (Ref. 30), we put the following definitions concerning the characteristics of consumption externalities.

Definition 2.1. *We say that consumption externalities indicate*

1. *jealousy if* $\partial u_i/\partial \overline{c}_i < 0$ $(\alpha > 0)$ *and admiration if* $\partial u_i/\partial \overline{c}_i > 0$ $(\alpha < 0)$.
2. *"keeping up with the Joneses" (KUJ) if* $\partial^2 u_i/\partial c_i \partial \overline{c}_i > 0$ $(\alpha(1-\eta) > 0)$ *and "running away from the Jones" (RAJ) if* $\partial^2 u_i/\partial c_i \partial \overline{c}_i < 0$ $(\alpha(1-\eta) < 0)$.

The definitions of jealousy and admiration are intuitively interpreted as follows. If the utility of an agent decreases as others' consumption rises, we can state that preferences exhibit jealousy. An example that leads to this situation is pollution which is increasing in consumption: the utility of a country decreases as other countries' consumption increases. On the other hand, if the utility increases as others' consumption increases, we state that preferences exhibit admiration. We can easily imagine several situations in which this form of utility function is rationalized. Consider, for example, international public goods, such as peace keeping, public health, or environmental preservation, the role of which is emphasized by Kindleberger (Ref. 31). If the international public goods are financed by consumption tax, the utility of a country is an increasing function of the tax revenues of other countries, that is, an increasing function of other countries' consumption.

Similarly, if the marginal utility increases as others' consumption rises, we state that preferences exhibit KUJ, and, if not we state that preferences exhibit RAJ. Here, note that, by the specification of our instantaneous utility function, our model fully and separately specifies the jealousy/admiration and KUJ/RAJ characteristics of consumption externalities, and thus, it has significant differences from the literature: Wirl and Feichtinger (Ref. 28) treat only the case that $\partial^2 u_i/\partial c_i \partial \overline{c_i} = 0$ and Futagami and Shinkai (Ref. 29) do only the case that $\operatorname{sign}\partial^2 u_i/\partial c_i \partial \overline{c_i} = \operatorname{sign}\partial u_i/\partial \overline{c_i}$.

The production technology has the Ak form, and therefore, the dynamics of the common capital stock is

$$\dot{k} = Ak - \sum_{j=1}^{N} c_j, \quad \text{given } k_0, \tag{2}$$

where $k : \mathbb{R}_+ \to \mathbb{R}_+$, $k_0 \in \mathbb{R}_{++}$, and $A \in \mathbb{R}_{++}$. Here, k denotes the stock of common capital, k_0 is its initial value, and A is a constant productivity parameter. If each agent chooses a symmetric strategy, we have $c = c_i$ for all $i = 1, \cdots, N$. In this case, from (2), the growth rate g can be written as

$$g = N(\omega_{\mathrm{SS}} - \omega), \tag{3}$$

where $\omega_{\mathrm{SS}} = A/N$ and $\omega = c/k$. Throughout this paper, we assume the following condition.

Assumption 2.1. *A, α, ρ, N, and ψ satisfy*

$$\psi < \frac{\rho}{A} < \min\left(1, \frac{1-\alpha}{N}\right), \tag{4}$$

where $\psi = (1-\alpha)(1 - 1/\eta)$.

Assumption 2.1 ensures that the problem is well defined, since the value function of each agent is well defined by Assumption 2.1. Moreover, this also ensures the rate of balanced growth is positive. If the value function did not have a finite value, the other concepts of optimality should be required: e.g., Stern (Ref. 32); Seierstad and Sydsæter (Ref. 33); Dockner, Jorgensen, van Long, and Sorger (Ref. 34).

3 Open-Loop Nash Equilibrium

We first consider a situation where agents commit themselves to their announced actions.

Definition 3.1. *An N-tuple of consumption paths, $(c_1, \ldots, c_N)$, is called an open-loop Nash equilibrium if, for each $i = 1, \ldots, N$, the c_i maximize (1) subject to (2).*

An open-loop Nash equilibrium is a plausible equilibrium concept in the situation where agents cannot observe the value of common capital at each time, and thus, condition their strategies only on the basis of the initial value of common capital, and precommit themselves to their future consumption paths. For example, this kind of the situation may occur when agents have to pay high cost to observe the current amount of the common capital, thereby they give up observing the current amount of infrastructure and adopt the open-loop Nash strategy.

To solve the problem, we define the following current value Hamiltonian:

$$\mathscr{H}_i = \frac{\eta}{\eta - 1}\left(c_i \overline{c}_i^{-\alpha}\right)^{1-\frac{1}{\eta}} + q_i^o\left(Ak - \sum_{j=1}^{N} c_j\right),$$

where q_i^o is the costate variable of the common capital. Here, note that the Hamiltonian is concave with respect to c_i and k under Assumption 2.1, and thus, the following conditions are necessary and sufficient conditions for maximization. The optimality conditions for this problem are given as

$$c_i^{-\frac{1}{\eta}} \overline{c}_i^{-\alpha\left(1-\frac{1}{\eta}\right)} = q_i^o, \tag{5}$$

$$Aq_i^o = \rho q_i^o - \dot{q}_i^o, \tag{6}$$

$$\lim_{t\to\infty} q_i^o k \exp(-\rho t) = 0. \tag{7}$$

At equilibrium, it follows from (5) and (6) that $c = \overline{c}_i = c_i$ for all $i = 1, \cdots, N$; thus, the Euler equation is given as

$$\frac{\dot{c}}{c} = \frac{A - \rho}{1 - \psi}.$$

Therefore, we have the following result.

Proposition 3.1. *The growth rate in the open-loop Nash equilibrium is given by*

$$g^o = \frac{A - \rho}{1 - \psi}. \tag{8}$$

Proof. From (3), (5), (6), we obtain the balanced growth rate in this open-loop Nash equilibrium. □

Note that, from conditions (5) to (7), we have a unique open-loop Nash equilibrium in this model. It is also easy to show that the open-loop Nash equilibrium is Pareto efficient. This efficiency result is similar to that derived by Chiarella, Kemp, van Long, and Okuguchi (Ref. 35), who show, contrary to the traditional view, that an open-loop Nash equilibrium can be socially efficient in various situations.

Finally, we put the following corollary to clarify the effects of the consumption externalities on the growth rate in the open-loop Nash equilibrium.

Corollary 3.1. *The growth rate in the open-loop Nash equilibrium is increasing in α if preferences exhibit jealousy and KUJ or if admiration and RAJ.*

Here, it should be noted that the above corollary is given only if all aspects of consumption externalities are fully specified.

4 Feedback Nash Equilibrium

We next consider the case where each agent does not commit himself to his future actions. This case can be analyzed by applying the feedback Nash equilibrium concept, which allows agents to choose and expect optimal consumption paths that depend on the current stock of common capital at each time.

Definition 4.1. *An N-tuple of consumption paths, $(c_1, \ldots, c_N)$, is called a feedback Nash equilibrium if, for each $i = 1, \ldots, N$, the c_i maximize (1) subject to (2) and $c_j = c_j(k)$ for $j \neq i$.*

Note that the feedback Nash equilibrium implies that agents can monitor the stock of common capital, and they condition their future consumption paths on the value of the common capital at each time. We solve the feedback Nash equilibrium by using dynamic programming. To solve the problem, we first define the value function of agent i. Given the stock of common capital k and the consumption path of agent i in the feedback Nash equilibrium (c_i^*), the value function $U_i(k)$ is given as

$$U_i(k) = \int_t^{\infty} \frac{\eta}{\eta - 1} \left(c_i^* \bar{c}_i(k)^{-\alpha}\right)^{1-\frac{1}{\eta}} \exp\left(-\rho(s-t)\right) \mathrm{d}s, \tag{9}$$

where $\bar{c}_i(k) = \sum_{j \neq i} c_j(k)/(N-1)$ and $c_j(k)$ is the feedback strategy of agent j. Hence, if the value function is continuously differentiable, the Hamilton-Jacobi-Bellman equation of this problem is given as

$$\rho U_i(k) = \max_{c_i} \left[\frac{\eta}{\eta - 1} \left(c_i \bar{c}_i(k)^{-\alpha}\right)^{1-\frac{1}{\eta}} + q_i^f \left(Ak - \sum_{j=1}^{N} c_j\right)\right], \tag{10}$$

where $q_i^f = dU_i/dk$. From (10), the first-order condition is obtained as

$$c_i^{-\frac{1}{\eta}} \bar{c}_i(k)^{-\alpha\left(1-\frac{1}{\eta}\right)} = q_i^f(k). \tag{11}$$

In what follows, we consider the equilibrium in linear and nonlinear strategies.

4.1 Equilibrium in Linear Strategies

In this subsection, we consider equilibrium where the agents adopt the following symmetric linear strategy.

$$c(k) = \beta k + \gamma, \tag{12}$$

where β and γ are constants. In this situation, we have the following lemma:

Lemma 4.1. *If agents adopt the linear strategy defined in (12), it holds that, for any $k \in \mathbb{R}_+$,*

$$\omega_{BGP} = \frac{\rho - \psi A}{1 - \alpha - \psi N} \tag{13}$$

and

$$U(k) = \frac{1}{\psi} \omega_{BGP}^{\psi - 1} k^{\psi}, \tag{14}$$

where ω_{BGP} denotes the ratio of consumption to common capital in the case that the agents adopt the linear strategy.

Proof. From (11), it holds that, at equilibrium,

$$U(k) = \frac{1}{\beta\psi} c(k)^{\psi} + \delta, \tag{15}$$

where δ denotes an integral constant. By using (11) and (15), (9) can be rewritten as

$$\frac{\rho}{\beta\psi} c(k)^{\psi} + \delta = \left(\frac{\eta}{\eta - 1} - N + A\frac{k}{\beta k + \gamma}\right) c(k)^{\psi}.$$

Therefore, for the above equality to hold for any value of k, it must hold that

$$\delta = \gamma = 0 \quad \text{and} \quad \beta = \frac{\rho - \psi A}{1 - \alpha - \psi N}. \tag{16}$$

Thus, by substituting (16) into (12) and (15), we have (13) and (14), respectively. □

From Lemma 4.1, we have the following proposition.

Proposition 4.1. *There exists a linear feedback Nash equilibrium, where agents adopt the linear strategy defined in (12) and the growth rate in the feedback Nash equilibrium is positive and given as*

$$g^f_{BGP} = N\left(\omega_{SS} - \omega_{BGP}\right), \tag{17}$$

where g^f_{BGP} *denotes the growth rate of the economy.*

Proof. If agents adopt (12), Lemma 4.1 holds. Therefore, substituting (13) into (3), we get

$$\frac{\dot{k}}{k} = \frac{\dot{c}}{c} = A + \frac{N\left(\psi A - \rho\right)}{1 - \alpha - \psi N} = N\left(\omega_{\mathrm{SS}} - \omega_{\mathrm{BGP}}\right),$$

where $\omega_{\mathrm{SS}} > \omega_{\mathrm{BGP}}$ by Assumption 2.1. Next, we check that (17) satisfies the terminal condition. From (14) in Lemma 4.1 and (17), we get

$$U(k) = \frac{1}{\psi}\omega_{\mathrm{BGP}}^{\psi-1}k_0^{\psi} \exp\left(\frac{\psi\left[A(1-\alpha) - \rho N\right]}{1 - \alpha - \psi N}t\right).$$

Therefore, we have

$$\lim_{t\to\infty} U(k) \exp\left(-\rho t\right) = \lim_{t\to\infty} \frac{1}{\psi}\omega_{\mathrm{BGP}}^{\psi-1}k_0^{\psi} \exp\left((\alpha - 1)\omega_{\mathrm{BGP}}t\right) = 0$$

since $\alpha < 1$ and $\omega_{\mathrm{BGP}} > 0$ by the assumptions. Finally, it is easily checked from (14) in Lemma 4.1 that the value function is continuously differentiable and the Hamilton-Jacobi-Bellman equation (10) is well defined. This implies that (14) actually gives the maximum value of the problem.[1] □

We put the following corollary to clarify the effects of the consumption externalities on the growth rate in the feedback Nash equilibrium.

Corollary 4.1. *The growth rate in the feedback Nash equilibrium is increasing in* α *if preferences exhibit jealousy and KUJ or if admiration and RAJ.*

Again, it should be noted that the above corollary is obtained only when all aspects of consumption externalities are fully specified.

[1] See Theorem 4.1 in Dockner et al. (Ref. 34).

4.2 Equilibrium in Nonlinear Strategies

This subsection considers a more general case where agents adopt symmetric nonlinear strategies. As in the literature, e.g. Tsutsui and Mino (Ref. 36), Dockner and Sorger (Ref. 8), Sorger (Ref. 9), Vencatachellum (Ref. 11), and Itaya and Shimomura (Ref. 37), our model also has a continuum of feedback Nash equilibria with nonlinear strategies. However, the property of the equilibrium in our model is in contrast with that of the literature.

Lemma 4.2. *In a symmetric feedback Nash equilibrium, it holds that*

$$k = Zc^{\frac{A}{A-\rho}(1-\psi)} + \frac{c}{\omega_{BGP}}, \tag{18}$$

where $Z \in \mathbb{R}$ denotes an integral constant.

Proof. In a symmetric equilibrium, it follows from (11) that (10) can be written as

$$\rho U = \left(\frac{\eta}{\eta-1} - N\right) q^{f\frac{\psi}{\psi-1}} + q^f Ak. \tag{19}$$

In light of (4), note that the coefficient of the first term on the right-hand side of the above equation is positive. Therefore, since (19) is the D'Alembert (or Lagrange) equation, the solution of (19) satisfies the following equation:

$$k = Zq^{f-\frac{A}{A-\rho}} + \frac{1-\alpha-\psi N}{\rho-\psi A} q^{f\frac{1}{\psi-1}}. \tag{20}$$

Again, using (11), we get (18). Here, it should be noted that, by its construction, the consumption paths derived from (20) guarantee the existence and continuous differentiability of q^f on a plausible domain. Therefore, as in Tsutsui and Mino (Ref. 36), the sufficient conditions for optimality is satisfied on the plausible domain since it implies that k is absolutely continuous on $[0, \infty)$ and U is continuously differentiable.[2] □

The sign of Z in Lemma 4.2 plays a crucial role in characterizing the dynamics of the economy, as will be clear in the statement in the last part of this section. Lemma 4.2 gives the dynamics of the economy as follows.

Proposition 4.2. *The dynamics of consumption in a symmetric feedback Nash equilibrium is characterized by (3) and*

$$\frac{\dot{c}}{c} = \frac{N(A-\rho)(\omega_{SS}-\omega)}{(1-\alpha-\psi N)(\omega_{UB}-\omega)}, \tag{21}$$

[2] See Theorem 4.1 in Dockner et al. (Ref. 34) for the sufficient condition for optimality.

where

$$\omega_{\mathrm{UB}} = \frac{(1-\psi)A}{1-\alpha-\psi N}.$$

Proof. The equation $\omega = \omega_{\mathrm{UB}}$ is not feasible, since it violates the differentiability of c; hereby we assume that $\omega \neq \omega_{\mathrm{UB}}$ in the following analyses. From Lemma 4.2, in the symmetric feedback Nash equilibrium, we have (18). Rearranging (18) yields

$$Z = \left(k - \frac{c}{\omega_{\mathrm{BGP}}}\right) c^{\frac{A}{A-\rho}(\psi-1)}.$$

Taking the logarithm of both sides of the above equation and differentiating them with respect to t, we have

$$\frac{\dot{k}}{k} = (\omega_{\mathrm{UB}} - \omega)\frac{1-\alpha-\psi N}{A-\rho}\frac{\dot{c}}{c}.$$

Therefore, by substituting (3) into the above equation, we have (21). □

Note that (21) is a necessary condition for the symmetric feedback Nash equilibrium, since it is derived from the first-order condition (11). Therefore, we need to check that the consumption path characterized by (21) satisfies the terminal condition to ensure it is optimal. The following proposition verifies that the consumption paths characterized by (21) satisfy the terminal condition. Before we state Proposition 4.3, we put the following lemma.

Lemma 4.3. *In the symmetric feedback Nash equilibrium, it holds that*

$$\dot{\omega} = \frac{N(\omega_{BGP} - \omega)(\omega - \omega_{SS})\,\omega}{\omega_{UB} - \omega}. \tag{22}$$

Proof. Since $\omega = \omega_{\mathrm{UB}}$ is not feasible, we assume here that $\omega \neq \omega_{\mathrm{UB}}$. Subtracting (3) from (21), we have

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} = \frac{N(\omega_{\mathrm{BGP}} - \omega)(\omega - \omega_{\mathrm{SS}})}{\omega_{\mathrm{UB}} - \omega}.$$

The above equation completes the proof. □

Therefore, the dynamics of the economy is the same as that of ω. Denote the growth rates of consumption and common capital in feedback Nash equilibrium by g_c^f and g_k^f, respectively. Then, we have the following proposition.

Proposition 4.3.

(i) *In the case $\omega_0 = \omega_{BGP}$, the growth rate in the feedback Nash equilibrium is given by (17).*

(ii) *In the case $\omega_0 > \omega_{\text{BGP}}$, the feedback Nash equilibria converge to steady states if $1 - \alpha < N$: $g_c^f \to 0$ and $g_k^f \to 0$ as $t \to \infty$.*

(iii) *In the case $\omega < \omega_{\text{BGP}}$, the growth rates in the feedback Nash equilibrium converge to that in the open-loop Nash equilibrium (8): $g_c^f \to g^o$ and $g_k^f \to A$ as $t \to \infty$,*

where ω_0 denotes the initial ratio of c to k: $\omega = \omega_0$ at $t = 0$.

Proof. (i) $\omega_0 = \omega_{\text{BGP}}$. Substituting $\omega_0 = \omega_{\text{BGP}}$ into (3) and (21) in Proposition 4.2 gives the result.

(ii) $\omega_0 > \omega_{\text{BGP}}$. See Appendix.

(iii) $\omega_0 < \omega_{\text{BGP}}$. It follows from (22) that $\dot{\omega} < 0$ for all $\omega < \omega_{\text{BGP}}$ and that $\dot{\omega} = 0$ at $\omega = 0$. Since this implies $\omega \to 0$ as $t \to \infty$, equation (3) and Proposition 4.2 yield

$$\lim_{t\to\infty} g_k^f = N(\omega_{\text{SS}} - \omega) = A \quad \text{and} \quad \lim_{t\to\infty} g_c^f = \frac{N(A-\rho)\omega_{\text{SS}}}{(1-\alpha-\psi N)\omega_{\text{UB}}} = \frac{A-\rho}{1-\psi}.$$

The right-hand side of the second equation is the same as the growth rate in the open-loop Nash equilibrium. Therefore, it satisfies the terminal condition under Assumption 2.1.

□

Figure 1 is the phase diagram in the case that $1 - \alpha < N$. The solid curves in Figure 1 depict the loci of capital and consumption satisfying (18) corresponding to various values of ω_0 (or Z) in Proposition 4.2. The BGP line plots the locus of capital and consumption in the case that $\omega_0 = \omega_{\text{BGP}}$ ($Z = 0$), and the curves above and below the BGP line depict the loci in the case that $\omega_0 > \omega_{\text{BGP}}$ ($Z < 0$) and $\omega_0 < \omega_{\text{BGP}}$ ($Z > 0$), respectively. The SS line depicts the $\dot{k} = 0$ ($\omega = \omega_{\text{SS}}$) locus, and thus, $\dot{k} > 0$ when a path is located below the line and $\dot{k} < 0$ when it lies in the area above the line. The slope of the SS line is larger than that of the BGP line by Assumption 2.1. It should also be noted that the SS line in the case $\omega_0 > \omega_{\text{BGP}}$ lies in the area below the UB line. The UB line plots $\omega = \omega_{\text{UB}}$, at which the slope of the peaks of the curves drawn from (18) become infinite.

The case $\omega_0 = \omega_{\text{BGP}}$ corresponds to the balanced growth path. Note that a balanced growth is realized only when $\omega_0 = \omega_{\text{BGP}}$. Here, it should be noted that the growth rate in this case coincides with (17); thus, it satisfies the terminal condition.

In the case $\omega_0 > \omega_{\text{BGP}}$, since the paths in the area above the UB line move up in the north-west direction along the curves drawn from (18) and reach the vertical axis in a finite period, the equilibrium paths drawn from (18) must lie between the UB and BGP lines. Moreover, since the paths between the UB and SS lines move down in the south-west direction and the paths between the SS and BGP lines move up in the north-east direction along the curves, the equilibrium paths converge to the SS line. This implies that there are infinitely many feedback Nash equilibria which converge to corresponding steady states. Note that these equilibrium paths are not Markov perfect in the strict sense, since the domain of each path does not span the entire state space. Although there are some approaches that these paths is made Markov perfect by artificial operations (e.g. Itaya and Shimomura (Ref. 37) and Clemhout and Wan Jr. (Ref. 38)), our attention is mainly focused to the next case.

In the case $\omega_0 < \omega_{\mathrm{BGP}}$, the paths must be below the BGP line and move up in the north-east direction along the curves. Along these curves, the growth rates converge to the balanced growth path in the open-loop Nash equilibrium. Therefore, as in Shibata (Ref. 5), we find that there also exist infinitely many nonbalanced growth paths in the feedback Nash equilibrium.

5 Comparison of the Growth Rates in Open-Loop and Feedback Nash Equilibria

5.1 Linear Strategies

As Tornell and Velasco (Ref. 4) and Shibata (Ref. 5) illustrate, in an economy without consumption externalities, the balanced growth rate in the feedback Nash equilibrium is lower than that in the open-loop Nash equilibrium. In this section, we show that the existence of consumption externalities may destroy this relationship between the two equilibrium growth rates.

Theorem 5.1. *Under Assumption 2.1, it holds that*

$$g^f_{\mathrm{BGP}} > g^o$$

if and only if

$$\alpha < 1 - N. \tag{23}$$

Proof. To compare the growth rates in the open-loop and feedback Nash equilibria, presented respectively in (8) and (17), we calculate their difference:

$$g^f_{\mathrm{BGP}} - g^o = \frac{\omega_{\mathrm{BGP}}}{1-\psi}\left(1 - \alpha - N\right). \tag{24}$$

Since the first multiplying term is positive by Assumption 2.1, the growth rate in the feedback Nash equilibrium becomes higher than that in the open-loop Nash equilibrium if and only if (23) holds. □

Note that, under (23), we can always choose the value of η to satisfy (4); that is, we can prove that, in this model, there exists a situation in which (24) is positive. Since $N > 1$, (23) states that the growth rate in the feedback Nash equilibrium is higher than that in the open-loop Nash equilibrium if preferences exhibit admiration to other agents' consumption. This result shows that, although the growth rate in the feedback Nash equilibrium is lower than that in the open-loop Nash equilibrium as generally argued, the presence of strong admiration reverses the conclusion.

Intuitively, the presence of jealousy implies that the utility of agents decreases with an increase of the others' consumption. This suggests that the presence of jealousy is a factor raising their current consumption and reducing their contribution to the accumulation of common capital. On the other hand, the presence of admiration implies that their utility increases with increase of the others' consumption, suggesting that admiration is a factor raising contribution to the accumulation of common capital. It shows that the growth rate in the feedback Nash equilibrium, where agents can change their consumption on the basis of others' consumption, may be higher than that in the open-loop Nash equilibrium if the degree of admiration is strong enough. Alternatively, the presence of admiration can also be interpreted as the other source of the free-rider problem, since they can enjoy utility from others' consumption. In this sense, it can be said that the growth rate without commitment becomes higher than that with commitment if the free ride on others' consumption is stronger than that on the common capital accumulation.

Finally, we put the following proposition concerning KUJ and RAJ.

Proposition 5.1. *The relative magnitude of the two growth rates is irrelevant to whether preferences exhibit KUJ or RAJ.*

Proposition 5.1 follows from the fact that (23) does not include the intertemporal elasticity of substitution. It should be noted that the above proposition is also given only if all aspects of consumption externalities are fully specified.

5.2 Nonlinear Strategies

Finally, we compare the open-loop Nash equilibrium and the nonlinear feedback Nash equilibrium. It is obvious that the growth rate in the open-loop Nash equilibrium is higher than that in the feedback Nash equilibrium in the long run in the case that $\omega_0 > \omega_{\text{BGP}}$, where consumption and capital stock converge to a steady state. In the case that $\omega_0 < \omega_{\text{BGP}}$, the growth rate of common capital converges to A since ω converges to 0. We have the following theorem on the rate of consumption growth.

Theorem 5.2. *Suppose that* $\omega_0 < \omega_{BGP}$*. Then, there exist infinitely many nonlinear feedback Nash equilibria such that*

$$g_c^f > g^o \quad \text{and} \quad g_k^f > g^o, \quad \forall t \in \mathbb{R}_+$$

if and only if $\alpha < 1 - N$*. Here,* g^f *denotes the growth rate of common capital.*

Proof. Rearranging (21), we have

$$g_c^f = \frac{A-\rho}{1-\psi}\frac{(1-\psi)A-(1-\psi)N\omega}{(1-\psi)A-(1-\alpha-\psi N)\omega}.$$

The above equation implies that $g_c^f > g^o$ if and only if $(1-\psi)N < 1-\alpha-\psi N$, that is, $\alpha < 1 - N$.

The fact that $g_k^f > g^o$ follows from (3), (8), and the fact that $\omega_{\mathrm{BGP}} > \omega$ for all $t \in \mathbb{R}$, which is derived from the assumptions of the theorem and the third statement in the proof of Lemma 4.2. □

Therefore, we find the same condition for the ordinal relationship between growth rates to be destroyed as in the case that agents adopt linear feedback strategies, (23), even if they adopt nonlinear ones.

6 Conclusions

This paper introduced consumption externalities into a dynamic game model of common capital accumulation. Contrary to the usual argument that the growth rate in a feedback Nash equilibrium is lower than that in an open-loop Nash equilibrium, we showed that the growth rates in the feedback Nash equilibria under both linear and nonlinear strategies could be higher than that in the open-loop Nash equilibrium if consumption externalities exist: the standard relationship between the growth rates in the two equilibrium is modified if agents strongly admire others' consumption, while the conventional relationship is maintained between the two growth rates if the agents envy others' consumption. This modification is irrelevant to whether preferences exhibit KUJ or RAJ.

Appendix

Proof of Proposition 4.3 in the Case that $\omega_0 > \omega_{\mathrm{BGP}}$

We assume that $1 - \alpha < N$. In this case, $\omega_{\mathrm{UB}} > \omega_{\mathrm{SS}} > \omega_{\mathrm{BGP}}$ by Assumption 2.1. We investigate (22) by examining two cases.

Case 1. $\omega_0 > \omega_{\mathrm{UB}}$. In this case, it holds that

$$\dot{\omega} \geq \dot{\omega}_0 > 0,$$

and thus, it follows from (3) that

$$\frac{\dot{k}}{k} \leq A - N\omega_0 < 0.$$

Therefore, the value of k reaches 0 in finite time. Since this implies that (18) cannot hold for any $c \in \mathbb{R}_+$ at some finite time, we exclude this case as a feedback Nash equilibrium.

Case 2. $\omega_{\mathrm{UB}} > \omega_0 > \omega_{\mathrm{BGP}}$. In this case,

$$\dot{\omega} < 0 \quad \text{if } \omega_0 > \omega_{\mathrm{SS}}$$

and

$$\dot{\omega} > 0 \quad \text{if } \omega_0 < \omega_{\mathrm{SS}}.$$

Therefore, $\omega \to \omega_{\mathrm{SS}}$ as $t \to \infty$, where

$$\dot{\omega} = \dot{k} = \dot{c} = 0.$$

The terminal condition is obviously satisfied in this case.

References

1. H.S. Gordon. The Economic Theory of a Common Property Resource. *Journal of Political Economy*, 62(2):124–142, 1954.

2. C. Fershtman and S. Nitzan. Dynamic Voluntary Provision of Public Goods. *European Economic Review*, 35:1057–1067, 1991.

3. D. Levhari and L.J. Mirman. The Great Fish War: An Example Using a Dynamic Cournot-Nash Solution. *The Bell Journal of Economics*, 11(1):322–334, 1980.

4. A. Tornell and A. Velasco. The Tragedy of the Commons and Economic Growth: Why Does Capital Flow from Poor to Rich Countries? *The Journal of Political Economy*, 100(6):1208–1231, 1992.

5. A. Shibata. Strategic Interactions in a Growth Model with Infrastructure Capital. *Metroeconomica*, 53(4):434–460, 2002.

6. J. Benhabib and R. Radner. The Joint Exploitation of a Productive Asset: A Game-Theoretic Approach. *Economic Theory*, 2(2):155–190, 1992.

7. G. Cozzi. R&D Cooperation and Growth. *Journal of Economic Theory*, 86 (1):17–49, 1999.

8. E.J. Dockner and G. Sorger. Existence and Properties of Equilibria for a Dynamic Game on Productive Assets. *Journal of Economic Theory*, 71(1): 209–227, 1996.

9. G. Sorger. Markov-Perfect Nash Equilibria in a Class of Resource Games. *Economic Theory*, 11(1):79–100, 1998.

10. D. Vencatachellum. Endogenous Growth with Strategic Interactions. *Journal of Economic Dynamics and Control*, 23(2):233–254, 1998.

11. D. Vencatachellum. A Differential R&D Game: Implications for Knowledge-Based Growth Models. *Journal of Optimization Theory and Applications*, 96 (1):175–189, 1998.

12. E.J. Dockner and K. Nishimura. Capital Accumulation Games with a Nonconcave Production Function. *Journal of Economic Behavior and Organization*, 57(4):408–420, 2005.

13. S. Luckraz. Process Spillovers and Growth. *Journal of Optimization Theory and Applications*, 2008. online first.

14. T. Veblen. *The Theory of the Leisure Class*. Macmillan, New York, 1912.

15. J.S. Duesenberry. *Income, Saving, and the Theory of Consumer Behavior*. Harvard University Press, Cambridge, MA, 1949.

16. A.B. Abel. Asset Prices under Habit Formation and Catching Up with the Joneses. *The American Economic Review*, 80(2):38–42, 1990.

17. G.M. Constantinides. Habit Formation: A Resolution of the Equity Premium Puzzle. *The Journal of Political Economy*, 98(3):519–543, 1990.

18. J. Galí. Keeping Up with the Joneses: Consumption Externalities, Portfolio Choice, and Asset Prices. *Journal of Money, Credit & Banking*, 26(1):1–8, 1994.

19. W.F. Liu and S.J. Turnovsky. Consumption Externalities, Production Externalities, and Long-Run Macroeconomic Efficiency. *Journal of Public Economics*, 89(5–6):1097–1129, 2005.

20. L. Ljungqvist and H. Uhlig. Tax Policy and Aggregate Demand Management under Catching Up with the Joneses. *The American Economic Review*, 90(3): 356–366, 2000.

21. S.J. Turnovsky and G. Monteiro. Consumption Externalities, Production Externalities, and Efficient Capital Accumulation under Time Non-separable Preferences. *European Economic Review*, 51(2):479–504, 2007.

22. K. Mino. Growth and Bubbles with Consumption Externalities. *Japanese Economic Review*, 59(1):33–53, 2008.

23. G. Kosicki. A Test of the Relative Income Hypothesis. *Southern Economic Journal*, pages 422–434, 1987.

24. A.E. Clark and A.J. Oswald. Satisfaction and Comparison Income. *Journal of Public Economics*, 61(3):359–381, 1996.

25. R.A. Easterlin. Will Raising the Incomes of All Increase the Happiness of All? *Journal of Economic Behavior and Organization*, 27(1):35–47, 1995.

26. J.H. Kagel, C. Kim, and D. Moser. Fairness in Ultimatum Games with Asymmetric Information and Asymmetric Payoffs. *Games and Economic Behavior*, 13(1):100–110, 1996.

27. D.J. Zizzo and A.J. Oswald. Are People Willing to Pay to Reduce Others' Incomes? *Annales d'Economie et de Statistique*, pages 39–65, 2001.

28. F. Wirl and G. Feichtinger. Intrafamiliar Consumption and Saving under Altruism and Wealth Considerations. *Economica*, 69(273):93–111, 2002.

29. K. Futagami and T. Shinkai. A Dynamic Game of Jealousy and Sympathy, 1996. *mimeo*, Ritsumeikan University.

30. B. Dupor and W.F. Liu. Jealousy and Equilibrium Overconsumption. *The American Economic Review*, 93(1):423–428, 2003.

31. C.P. Kindleberger. International Public Goods without International Government. *The American Economic Review*, pages 1–13, 1986.

32. Le Stern. Criteria of Optimality in the Infinite-Time Optimal Control Problem. *Journal of Optimization Theory and Applications*, 44(3):497–508, 1984.

33. A. Seierstad and K. Sydsæter. *Optimal Control Theory with Economic Applications*. Advanced Textbooks in Economics. North Holland, 3 edition, 1987.

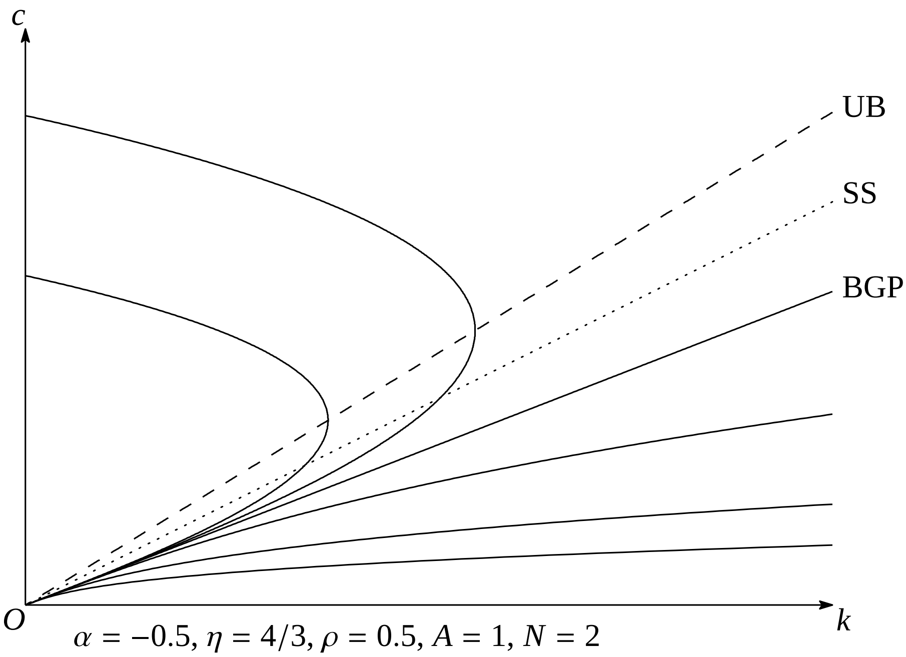
34. E.J. Dockner, S. Jorgensen, N. van Long, and G. Sorger. *Differential Games in Economics and Management Science*. Cambridge University Press, 2000.

35. C. Chiarella, M.C. Kemp, N. van Long, and K. Okuguchi. On the Economics of International Fisheries. *International Economic Review*, 25(1): 85–92, 1984.

36. S. Tsutsui and K. Mino. Nonlinear Strategies in Dynamic Duopolistic Competition with Sticky Prices. *Journal of Economic Theory*, 52(1):136–161, 1990.

37. J. Itaya and K. Shimomura. A Dynamic Conjectural Variations Model in the Private Provision of Public Goods: A Differential Game Approach. *Journal of Public Economics*, 81(1):153–172, 2001.

38. S. Clemhout and H.Y. Wan Jr. Endogenous Growth as a Dynamic Game. In G.J. Olsder, editor, *New Trends in Dynamic Games and Applications*. Birkhauser, Boston, MA, 1996.



$\alpha = -0.5$, $\eta = 4/3$, $\rho = 0.5$, $A = 1$, $N = 2$

BGP: $c = \frac{\rho - \psi A}{1 - \alpha - \psi N} k$, SS: $c = \omega_{\mathrm{SS}} k$, UB: $c = \omega_{\mathrm{UB}} k$

Figure 1: Dynamics for feedback Nash equilibrium